

PROBABILISTIC STAR DISCREPANCY BOUNDS FOR LACUNARY POINT SETS

THOMAS LÖBBE ¹

ABSTRACT. By a result of Heinrich, Novak, Wasilkowski and Woźniakowski the inverse of the star discrepancy $n(d, \varepsilon)$ satisfies $n(d, \varepsilon) \leq c_{\text{abs}} d \varepsilon^{-2}$. Equivalently for any N and d there exists a set of N points in $[0, 1]^d$ with star discrepancy bounded by $\sqrt{c_{\text{abs}} \cdot d/N}$. They actually proved that a set of independent uniformly distributed random points satisfies this upper bound with positive probability. Although Aistleitner and Hofer later refined this result by proving a precise value of c_{abs} depending on the probability with which the inequality holds, so far there is no general construction for such a set of points known. In this paper we consider the sequence $(x_n)_{n \geq 1} = (\langle 2^{n-1} x_1 \rangle)_{n \geq 1}$ for a uniformly distributed point $x_1 \in [0, 1]^d$ and prove that the star discrepancy is bounded by $C \sqrt{d \log_2 d/N}$. The precise value of C depends on the probability with which this upper bound holds.

1 Introduction

A sequence of vectors $(x_n)_{n \geq 1} = (x_{n,1}, \dots, x_{n,d})_{n \geq 1}$ of real numbers in $[0, 1]^d$ is called *uniformly distributed modulo one* if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{\mathcal{A}}(x_n) = \lambda(\mathcal{A}) \quad (1.1)$$

for any axis-parallel box $\mathcal{A} \subset [0, 1]^d$ where $\mathbf{1}_{\mathcal{A}}$ denotes the indicator function on the set \mathcal{A} and λ denotes the Lebesgue-measure on $[0, 1]^d$. The star discrepancy of the first N elements of $(x_n)_{n \geq 1}$ is defined by

$$D_N^*(x_1, \dots, x_N) = \sup_{\mathcal{A} \in \mathcal{B}^*} \left| \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{\mathcal{A}}(x_n) - \lambda(\mathcal{A}) \right|, \quad (1.2)$$

where \mathcal{B}^* denotes the set of all axis-parallel boxes $\mathcal{A} = \prod_{i=1}^d [0, \beta_i) \subset [0, 1]^d$ with one corner in 0. A sequence of points $(x_n)_{n \geq 1}$ is called a *low-discrepancy sequence* if

$$D_N^*(x_1, \dots, x_N) \leq C \frac{(\log N)^d}{N} \quad (1.3)$$

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for all $N \geq 1$ and some absolute constant $C > 0$. Furthermore Roth [8] showed that there exists a constant C_d depending only on d such that for any sequences $(x_n)_{n \geq 1}$ we have

$$D_N^*(x_1, \dots, x_N) \geq C_d \frac{(\log N)^{d/2}}{N}. \quad (1.4)$$

Thus the asymptotic behaviour of a low-discrepancy sequence is not far from optimal. Nevertheless, if N is small compared to d then the upper bound in (1.3) is not suitable. Therefore the *inverse of the star discrepancy* was introduced. Let $n(d, \varepsilon)$ denote the smallest number N such that there exists a N -element set of points in $[0, 1)^d$ such that the star discrepancy of this point set is bounded by ε . In 2001 Heinrich, Novak, Wasilkowski and Woźniakowski [6] proved that

$$n(d, \varepsilon) \leq c_{\text{abs}} d \varepsilon^{-2} \quad (1.5)$$

holds for all $d \geq 1$ and $\varepsilon > 0$ with some absolute constant $c_{\text{abs}} > 0$. On the other hand Hinrichs [7] showed

$$n(d, \varepsilon) \geq c_{\text{abs}} d \varepsilon^{-1} \quad (1.6)$$

for all $d \geq 1$ and $\varepsilon > 0$ and some possibly different absolute constant $c_{\text{abs}} > 0$. Thus the inverse of the star discrepancy depends linearly on the dimension, only the precise dependence on ε is still unknown. By (3.21) there exists a set of N points in $[0, 1)^d$ with

$$D_N^*(x_1, \dots, x_N) \leq \sqrt{c_{\text{abs}}} \sqrt{\frac{d}{N}}. \quad (1.7)$$

In fact, Heinrich et al. proved that a set of independent uniformly distributed random points, i.e. a Monte Carlo point set, satisfies (1.7) with positive probability. This result was later refined by Aistleitner and Hofer [2] who gave an upper bound on c_{abs} depending on the probability with which (1.7) is satisfied. Although they showed that even for moderate constants the inequality holds with high probability so far there is no general construction of a suitable point set known.

For a uniformly distributed point $x_1 \in [0, 1)^d$ let $(x_n)_{n \geq 1}$ be a sequence with $x_{n+1} = (x_{n+1,i})_{i=1,\dots,d} = (\langle 2x_{n,i} \rangle)_{i=1,\dots,d}$ for all $n \geq 1$ where $\langle \cdot \rangle$ denotes the fractional part of a rational number. Conze, Le Borgne and Roger [3] proved that a system of random variables $(f(x_n))_{n \geq 1}$ where $f : [0, 1)^d \rightarrow \mathbb{R}$ is a centered indicator function on a box satisfies the Central Limit Theorem. Thus the asymptotic behaviour of this sequence which is a particular example of a lacunary system $(f(M_n x))_{n \geq 1}$ which in general is defined by a centered one-periodic function f with "nice" analytic properties and a fast growing sequence of $d \times d$ integer valued matrices satisfying a Hadamard gap condition

$$\|M_{n+k}^T j\|_\infty \geq q^k \|M_n^T j\|_\infty \quad (1.8)$$

for all $n, k \geq 1$, $j \in \mathbb{Z}^d$ with $0 < \log_q \|j\|_\infty \leq k$ and some absolute constant $q > 1$ is similar to the behaviour of independent random variables.

The number of digits which are necessary to simulate N points of this sequence with H digits precision is of order $\mathcal{O}(d(H + N))$ and thus is much smaller than the number

of digits to simulate N independent random points which is $\mathcal{O}(dHN)$. Therefore we consider this randomized sequence $(x_n)_{n \geq 1}$. We prove an upper bound on the star discrepancy which holds with high probability. Compared to (1.7) this upper bound has up to some constant only an additional $\sqrt{\log_2 d}$ -factor. Our main result is stated in the following

Theorem 1.1 *Let $N \geq 1$ and $d \geq 2$ be integers.*

Then for any $0 < \varepsilon < 1$ the star discrepancy of the point set (x_1, \dots, x_N) satisfies

$$D_N^*(x_1, \dots, x_N) \leq (87 - 7d^{-1} \log \varepsilon) \sqrt{\frac{d \log_2 d}{N}}$$

with probability at least $1 - \varepsilon$.

2 Preliminaries

Lemma 2.1 (Maximal Bernstein inequality, [4, Lemma 2.2]) *For an integer $N \geq 1$ let Z_1, \dots, Z_N be a sequence of i.i.d. random variables with mean zero and variance $\sigma^2 > 0$ such that $|Z_1| \leq 1$. Then for any $t > 0$ we have*

$$\mathbb{P} \left(\max_{M \in \{1, \dots, N\}} \left| \sum_{n=1}^M Z_n \right| > t \right) \leq 2 \exp \left(-\frac{t^2}{2N\sigma^2 + 2t/3} \right). \quad (2.1)$$

Let $v, w \in [0, 1)^d$. We write $v \leq w$ if $v_i \leq w_i$ for all $i \in \{1, \dots, d\}$. For some $\delta > 0$ a set Δ of elements in $[0, 1)^d \times [0, 1)^d$ is called a δ -bracketing cover if for every $x \in [0, 1)^d$ there exists $(v, w) \in \Delta$ with $v \leq x \leq w$ and $\lambda(\overline{[v, w]}) \leq \delta$ for $\overline{[v, w]} = [0, w) \setminus [0, v)$. The following Lemma gives an upper bound on the cardinality of a δ -bracketing cover.

Lemma 2.2 ([5, Theorem 1.15]) *For any $d \geq 1$ and $\delta > 0$ there exists some δ -bracketing cover Δ with*

$$|\Delta| \leq \frac{1}{2} (2e)^d (\delta^{-1} + 1)^d.$$

Corollary 2.3 *For any integers $d \geq 1$ and $h \geq 1$ there exists a 2^{-h} -bracketing cover Δ with*

$$|\Delta| \leq \frac{1}{2} (2e)^d (2^{h+2} + 1)^d$$

such that for any $(v, w) \in \Delta$ and any $i \in \{1, \dots, d\}$ we have

$$\begin{aligned} v_i &= 2^{-(h+1+\lceil \log_2 d \rceil)} a_i, \\ w_i &= 2^{-(h+2+\lceil \log_2 d \rceil)} b_i \end{aligned}$$

for some integers $a_i \in \{0, 1, \dots, 2^{h+1+\lceil \log_2 d \rceil}\}$ and $b_i \in \{0, 1, \dots, 2^{h+2+\lceil \log_2 d \rceil}\}$.

Proof. Let Δ be some $2^{-(h+2)}$ -bracketing cover of $[0, 1]^d$. By Lemma 2.2 we have

$$|\Delta| \leq \frac{1}{2}(2e)^d(2^{(h+2)} + 1)^d.$$

For $(v, w) \in \Delta$ and $i \in \{1, \dots, d\}$ define

$$\begin{aligned} y_{v,i} &= \max \left\{ 2^{-(h+1+\lceil \log_2 d \rceil)} a_i \leq v_i : a_i \in \mathbb{Z} \right\}, \\ z_{w,i} &= \min \left\{ 2^{-(h+2+\lceil \log_2 d \rceil)} b_i \geq w_i : b_i \in \mathbb{Z} \right\}. \end{aligned}$$

For $y_v = (y_{v,i})_{i \in \{1, \dots, d\}} \in [0, 1]^d$ we obtain

$$\lambda(\overline{[y_v, v]}) \leq \sum_{i=1}^d 2^{-(h+1+\lceil \log_2 d \rceil)} \leq 2^{-(h+1)}.$$

Analogously for $z_w = (z_{w,i})_{i \in \{1, \dots, d\}} \in [0, 1]^d$ we have

$$\lambda(\overline{[z, z_w]}) \leq 2^{-(h+2)}.$$

Thus we get

$$\lambda(\overline{[y_v, z_w]}) \leq \lambda(\overline{[y_v, v]}) + \lambda(\overline{[v, w]}) + \lambda(\overline{[w, z_w]}) \leq 2^{-h}.$$

Set $\tilde{\Delta} = \{(y_v, z_w) : (v, w) \in \Delta\}$. Since Δ is a $2^{-(h+2)}$ -bracketing cover for any $x \in [0, 1]^d$ there exists $(v, w) \in \Delta$ and $(y_v, z_w) \in \tilde{\Delta}$ with $y_v \leq v \leq x \leq w \leq z_w$. Therefore $\tilde{\Delta}$ is a 2^{-h} -bracketing cover and the conclusion of the proof follows by $|\tilde{\Delta}| \leq |\Delta|$.

3 Proof of main theorem

The proof of this Theorem is mainly based on [1]. For some integers $N \geq 1$ and $d \geq 1$ we simply write

$$D_N^d(x_{n,i}) = D_N^d((x_{1,1}, \dots, x_{1,d}), \dots, (x_{N,1}, \dots, x_{N,d})).$$

For $N \geq 1$ and $d \geq 1$ set

$$H = \left\lceil \frac{\log_2 N}{2} - \frac{\log_2(d \log_2 d)}{2} - 2 \right\rceil. \quad (3.1)$$

As a consequence for any $h \in \{0, \dots, H\}$ we have

$$\sqrt{d \log_2 d} \sqrt{N} \leq 2^{-h} N. \quad (3.2)$$

For any $h \in \{1, \dots, H\}$ let Δ_h be a 2^{-h} -bracketing cover of $[0, 1]^d$. By Corollary 2.3 we may assume

$$|\Delta_h| \leq \frac{1}{2}(2e)^d(2^{h+2} + 1)^d. \quad (3.3)$$

For any $y \in [0, 1)^d$ we now define a finite sequence of points $\beta_h(y)$ for $h \in \{0, \dots, H+1\}$ in the following manner. Let $(v, w) \in \Delta_H$ be such that $v \leq y \leq w$. We set $\beta_{H+1}(y) = w$ and $\beta_H(y) = v$. The points $\beta_1(y), \dots, \beta_{H-1}(y)$ are defined by induction. Thus assume that for some $h \in \{1, \dots, H-1\}$ the point $\beta_{h+1}(y)$ is already defined. Let $(v, w) \in \Delta_h$ with $v \leq \beta_{h+1}(y) \leq w$ and set $\beta_h(y) = v$. Moreover set $\beta_0(y) = 0$. Therefore we observe

$$0 = \beta_0(y) \leq \beta_1(y) \leq \dots \leq \beta_H(y) \leq x \leq \beta_{H+1}(y) \leq 1.$$

For $h \in \{0, \dots, H-1\}$ we have $(\beta_h(y), w) \in \Delta_h$ for some point $w \in [0, 1)^d$. Furthermore we have $(\beta_H(y), \beta_{H+1}(y)) \in \Delta_H$. Then by Corollary 2.3 for $h \in \{0, \dots, H+1\}$ and $i \in \{1, \dots, d\}$ there exist integers $a_{h,i} \in \{0, \dots, 2^{h+1+\log_2 d}\}$ such that

$$(\beta_h(y))_i = 2^{-(h+1+\log_2 d)} a_{h,i}. \quad (3.4)$$

For $h \in \{0, \dots, H\}$ set $K_h(y) = \overline{[\beta_h(y), \beta_{h+1}(y))}$. Note that the sets $K_h(y)$ are pairwise disjoint and satisfy

$$\bigcup_{h=0}^{H-1} K_h(x) \subseteq [0, x] \subseteq \bigcup_{h=0}^H K_h(x) \quad (3.5)$$

By definition $\beta_h(y) \leq \beta_{h+1}(y) \leq w$ for some $w \in [0, 1)^d$ with $(\beta_h(y), w) \in \Delta_h$ and hence

$$\lambda(K_h(y)) \leq \lambda(\overline{[\beta_h(y), w)}) \leq 2^{-h} \quad (3.6)$$

for any $h \in \{0, \dots, H\}$. Now define

$$S_h = \left\{ \overline{[\beta_h(y), \beta_{h+1}(y))} : y \in [0, 1)^d \right\}.$$

Observe that we may define the points β_h such that $\beta_h(y) = \beta_h(z)$ for $y, z \in [0, 1)^d$ with $\beta_{h+1}(y) = \beta_{h+1}(z)$. Therefore by Corollary 2.3 we have

$$|S_h| = \left| \left\{ \beta_{h+1}(y) : y \in [0, 1)^d \right\} \right| \leq |\Delta_{h+1}| \leq \frac{1}{2} (2e)^d (\sqrt{5})^{(h+3)d} \quad (3.7)$$

for any integer $h \in \{0, \dots, H\}$. Note that hereafter we skip the point y in the notation of the points β_h and the sets K_h to simplify notations. Then by (3.5) we have

$$\sum_{n=1}^N \mathbf{1}_{[0,y)}(x_n) \geq \sum_{n=1}^N \mathbf{1}_{[0,\beta_H)}(x_n) = \sum_{h=0}^{H-1} \sum_{n=1}^N (\mathbf{1}_{K_h}(x_n) - \lambda(K_h)). \quad (3.8)$$

Analogously we also get

$$\sum_{n=1}^N \mathbf{1}_{[0,y)}(x_n) \leq \sum_{n=1}^N \mathbf{1}_{[0,\beta_{H+1})}(x_n) = \sum_{h=0}^H \sum_{n=1}^N (\mathbf{1}_{K_h}(x_n) - \lambda(K_h)). \quad (3.9)$$

By using Bernstein inequality we now shall give a lower bound on the probability that the inequality

$$\left| \sum_{n=1}^N (\mathbf{1}_{K_h}(x_n) - \lambda(K_h)) \right| > t \quad (3.10)$$

holds simultaneously for all $h \in \{0, \dots, H\}$ and some $t > 0$ to specified later. Observe that in general the random variables $f_{K_h}(x_n) = \mathbf{1}_{K_h}(x_n) - \lambda(K_h)$ are not independent. Thus we may not apply the Bernstein inequality directly. Therefore we decompose the set of numbers $\{1, \dots, N\}$ into several modulo classes. If the distance between two consecutive indices n_l, n_{l+1} in the same class is large enough, i.e. $n_{l+1} - n_l \geq h + 2 + \lceil \log_2 d \rceil$, the random variables are stochastically independent, i.e.

$$\mathbb{P}(f_{K_h}(x_{n_1}) = c_1, \dots, f_{K_h}(x_{n_k}) = c_k) = \prod_{l=1}^k \mathbb{P}(f_{K_h}(x_{n_l}) = c_l). \quad (3.11)$$

We only prove the case $k = 2$. The general case follows by induction.

By (3.4) the set K_h is a union of axis-parallel boxes such that each corner of any box is of the form

$$\left(2^{-(h+2+\lceil \log_2 d \rceil)} a_1, \dots, 2^{-(h+2+\lceil \log_2 d \rceil)} a_d\right) \quad (3.12)$$

such that $a_i \in \{0, 1, \dots, 2^{h+2+\lceil \log_2 d \rceil}\}$ for any $i \in \{1, \dots, d\}$. Furthermore let $n, n' \in \{1, \dots, N\}$ be two indices with $n' - n \geq h + 2 + \lceil \log_2 d \rceil$. We define a decomposition of $[0, 1]^d$ by

$$\Sigma = \left\{ \prod_{i=1}^d \left[2^{-(n'-1+\lceil \log_2 d \rceil)} a_i, 2^{-(n'-1+\lceil \log_2 d \rceil)} (a_i + 1) \right) : \right. \\ \left. a_i \in \{0, 1, \dots, 2^{n'-1+\lceil \log_2 d \rceil} - 1\}, i \in \{1, \dots, d\} \right\}.$$

Note that by (3.12) the function f_{K_h} is constant on any box $\mathcal{B} \in \Sigma$. For some $c_1 \in \mathbb{R}$ define

$$\Sigma_{c_1} = \{\mathcal{B} \in \Sigma : f_{K_h}(x_n) = c_1 \text{ for all } x_1 = (x_{1,1}, \dots, x_{1,d}) \in \mathcal{B}\}.$$

Since $x_{n',i} = 2^{n'-1} x_{1,i}$ for all $i \in \{1, \dots, d\}$ we have $f_{K_h}(x_{n'}) = f_{K_h}(x'_{n'})$ where $x'_{n'} = (x'_{n',1}, \dots, x'_{n',d})$ with $x'_{n',i} = 2^{n'-1} x'_{1,i}$ is an instance of the matrix for some initial value $x'_1 = (x'_{1,1}, \dots, x'_{1,d})$ with $x'_{1,i} = x_{1,i} + 2^{-(n'-1+\lceil \log_2 d \rceil)} a_i$ and $a_i \in \{0, 1, \dots, 2^{n'-1+\lceil \log_2 d \rceil} - 1\}$ for all $i \in \{1, \dots, d\}$. Therefore for any $c_2 \in \mathbb{R}$ and any $\mathcal{B}, \mathcal{B}' \in \Sigma$ we have

$$\mathbb{P}(f_{K_h}(x_{n'}) = c_2 | x_1 \in \mathcal{B}) = \mathbb{P}(f_{K_h}(x_{n'}) = c_2 | x_1 \in \mathcal{B}').$$

Hence for any $c_2 \in \mathbb{R}$ and any $\mathcal{B} \in \Sigma$ we get

$$\begin{aligned} \mathbb{P}(f_{K_h}(x_{n'}) = c_2) &= \sum_{\mathcal{B}' \in \Sigma} \mathbb{P}(f_{K_h}(x_{n'}) = c_2 | x_1 \in \mathcal{B}') \mathbb{P}(x_1 \in \mathcal{B}') \\ &= \mathbb{P}(f_{K_h}(x_{n'}) = c_2 | x_1 \in \mathcal{B}) \sum_{\mathcal{B}' \in \Sigma} \mathbb{P}(x_1 \in \mathcal{B}') \\ &= \mathbb{P}(f_{K_h}(x_{n'}) = c_2 | x_1 \in \mathcal{B}). \end{aligned}$$

Moreover for any $c_1, c_2 \in \mathbb{R}$ we obtain

$$\begin{aligned}
& \mathbb{P}(f_{K_h}(r_{n'}) = c_2 | f_{K_h}(x_n) = c_1) \\
&= \frac{\mathbb{P}(f_{K_h}(x_{n'}) = c_2, f_{K_h}(x_n) = c_1)}{\mathbb{P}(f_{K_h}(x_n) = c_1)} \\
&= \frac{\sum_{\mathcal{B} \in \Sigma} \mathbb{P}(f_{K_h}(x_{n'}) = c_2, f_{K_h}(x_n) = c_1 | x_1 \in \mathcal{B}) \mathbb{P}(x_1 \in \mathcal{B})}{\mathbb{P}(f_{K_h}(x_n) = c_1)} \\
&= \sum_{\mathcal{B} \in \Sigma_{c_1}} \mathbb{P}(f_{K_h}(x_{n'}) = c_2 | x_1 \in \mathcal{B}) \frac{\mathbb{P}(x_1 \in \mathcal{B})}{\mathbb{P}(f_{K_h}(x_n) = c_1)} \\
&= \mathbb{P}(f_{K_h}(x_{n'}) = c_2).
\end{aligned}$$

Thus (3.11) is proved. Set $\kappa = \kappa_h = \lceil \log_2(h + 2 + \lceil \log_2 d \rceil) \rceil$. Furthermore set $Q(N, \kappa, \gamma) = \{n \in \{1, \dots, N\} : n \equiv \gamma \pmod{2^\kappa}\}$.

Then for $h \in \{0, \dots, H\}$ by Lemma 2.1 we have

$$\begin{aligned}
& \mathbb{P}\left(\left|\sum_{n=1}^N (\mathbf{1}_{K_h}(x_n) - \lambda(K_h))\right| > t\right) \\
& \leq \sum_{\gamma=1}^{2^\kappa} \mathbb{P}\left(\left|\sum_{n \in Q(N, \kappa, \gamma)} \mathbf{1}_{K_h}(r_n) - \lambda(K_h)\right| > \frac{t}{2^\kappa}\right) \\
& \leq 2 \sum_{\gamma=1}^{2^\kappa} \exp\left(-\frac{t^2/2^{2\kappa}}{2\left(\sum_{n \in Q(N, \kappa, \gamma)} 1\right) \lambda(K_h)(1 - \lambda(K_h)) + 2t/(3 \cdot 2^\kappa)}\right) \quad (3.13) \\
& \leq 2^{\kappa+1} \exp\left(\frac{t^2/2^\kappa}{4N \cdot 2^{-h} + 2t/3}\right).
\end{aligned}$$

For $h \geq 1$ set $t = C_1 \sqrt{d \log_2 d} \sqrt{N} \sqrt{h \cdot 2^{-h}}$ for a constant $C_1 > 0$ to be specified later. By (3.2) we observe

$$\frac{2t}{3} \leq \frac{2}{\sqrt{3}} C_1 \log_2 d \cdot N \cdot 2^{-h}.$$

Thus we get

$$\frac{t^2}{4 \cdot 2^{-h} N + 2t/3} \geq \frac{C_1^2 d \log_2 d \cdot 2^{-h} h N}{4 \cdot 2^{-h} N + 2/\sqrt{3} \cdot C_1 \log_2 d \cdot 2^{-h} N} \geq \frac{C_1^2 d h}{4 + 2/\sqrt{3} \cdot C_1}.$$

Plugging this into (3.13) we obtain

$$\begin{aligned}
\mathbb{P} \left(\left| \sum_{n=1}^N (\mathbf{1}_{K_h}(x_n) - \lambda(K_h)) \right| > t \right) \\
\leq 2 \exp \left(\kappa \log 2 - \frac{C_1^2}{4 + 2/\sqrt{3} \cdot C_1} dh \right) \\
\leq 2 \exp \left(\lceil \log_2(h + 2 + \lceil \log_2 d \rceil) \rceil \log 2 - \frac{C_1^2}{4 + 2/\sqrt{3} \cdot C_1} dh \right) \quad (3.14) \\
\leq 2 \exp \left(- \left(\frac{C_1^2}{4 + 2/\sqrt{3} \cdot C_1} - 1 \right) dh \right)
\end{aligned}$$

where the last inequality holds for $d \geq 2, h \geq 1$. Similarly for $h = 0$ we set $t = C_2 \sqrt{d \log_2 d} \sqrt{N}$ for a constant $C_2 > 0$ to be specified later. Repeating the above calculation we show

$$\mathbb{P} \left(\left| \sum_{n=1}^N (\mathbf{1}_{K_h}(x_n) - \lambda(K_h)) \right| > t \right) \leq 2 \exp \left(- \left(\frac{C_2^2}{4 + 2/3 \cdot C_2} - 1 \right) d \right) \quad (3.15)$$

for $d \geq 2$. Now define

$$C_3 = \frac{C_1^2}{4 + 2/\sqrt{3} \cdot C_1} - 1, \quad C_4 = \frac{C_2^2}{4 + 2/3 \cdot C_2} - 1. \quad (3.16)$$

By (3.7) the statement of the Theorem immediately follows if we show

$$1 - \frac{1}{2} (2e)^d (\sqrt{5})^{3d} \cdot 2e^{-C_4 d} - \frac{1}{2} (2e)^d \sum_{h=1}^H (\sqrt{5})^{(h+3)d} \cdot 2e^{-C_3 dh} \geq 1 - \varepsilon. \quad (3.17)$$

Thus it is enough to choose constants C_3, C_4 large enough such that

$$\frac{1}{2} (2e)^d (\sqrt{5})^{3d} \cdot 2e^{-C_4 d} \leq \frac{\varepsilon}{2} \quad (3.18)$$

and

$$\frac{1}{2} (2e)^d (\sqrt{5})^{(h+3)d} \cdot 2e^{-C_3 dh} \leq \frac{\varepsilon}{2^{h+1}} \quad (3.19)$$

for all $h \in \{1, \dots, H\}$. Observe that (3.18) is satisfied for

$$C_4 = 4.46 - \frac{\log \varepsilon}{d} \geq 1 + \log 2 + 1.5 \log 5 + \frac{\log 2}{d} - \frac{\log \varepsilon}{d}. \quad (3.20)$$

Similarly (3.19) is equivalent to

$$(1 + 2 \log 2 + 1.5 \log 5) d + \frac{\log 5}{2} dh + \log 2 \cdot h - \log \varepsilon \leq C_3 dh.$$

Since $h \geq 1$ we may choose

$$C_3 = 6.31 - d^{-1}h^{-1} \log \varepsilon. \quad (3.21)$$

By (3.16) we may set

$$C_1 = 15.465 - 1.155d^{-1} \log \varepsilon, \quad (3.22)$$

$$C_2 = 9.864 - 2/3 \cdot d^{-1} \log \varepsilon. \quad (3.23)$$

Thus with probability at least $1 - \varepsilon$ by (3.8), (3.14) and (3.15) we have

$$\begin{aligned} \sum_{n=1}^N \mathbf{1}_{[0,y)}(x_n) &\leq \sum_{n=1}^N \sum_{h=1}^{H+1} \mathbf{1}_{K_h}(x_n) \\ &\leq \sum_{n=1}^N \left(\lambda([0, \beta_1)) + \left(9.864 - \frac{2 \log \varepsilon}{3d} \right) \sqrt{\frac{d \log_2 d}{N}} \right) \\ &\quad + \sum_{n=1}^N \sum_{h=1}^H \left(\lambda([\beta_h, \beta_{h+1})) + \left(15.465 - \frac{1.155 \log \varepsilon}{d} \right) \sqrt{\frac{d \log_2 d}{N}} \sqrt{2^{-h}h} \right) \\ &\leq \sum_{n=1}^N \left(\lambda([0, y)) + \lambda([y, \beta_{H+1})) + (82.357 - 6.081d^{-1} \log \varepsilon) \sqrt{\frac{d \log_2 d}{N}} \right) \\ &\leq \sum_{n=1}^N \lambda([0, y)) + \sum_{n=1}^N (86.357 - 6.081d^{-1} \log \varepsilon) \sqrt{\frac{d \log_2 d}{N}}. \end{aligned}$$

Analogously we obtain

$$\sum_{n=1}^N \mathbf{1}_{[0,y)}(x_n) \geq \sum_{n=1}^N \lambda([0, y)) - \sum_{n=1}^N (86.357 - 6.081d^{-1} \log \varepsilon) \sqrt{\frac{d \log_2 d}{N}}$$

with probability at least $1 - \varepsilon$.

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DEPT. OF MATHEMATICS, BIELEFELD UNIV., P.O.Box 100131, 33501 Bielefeld, Germany
E-Mail address: `tloebbe@math.uni-bielefeld.de`